A Set and Collection Lemma

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Abstract

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent. Let $\alpha(G)$ stand for the cardinality of a largest independent set.

In this paper we prove that if Λ is a n*on-em*pty collection of maximum independent sets of a graph G, and S is an independent set, then

- there is a matching from $S - \cap\Lambda$ into $\cup\Lambda - S$, and
- $|S| + \alpha(G) \leq |\cap\Lambda \cap S| + |\cup\Lambda \cup S|$.

Based on these findings we provide alternative proofs for a number of well-known lemmata, as the "*Maximum Stable Set Lemma*" due to Claude Berge and the "*Clique Collection Lemma*" due to András Hajnal.

Keywords: matching, independent set, stable set, core, corona, clique

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$, and we use $G - w$, whenever $W = \{w\}$.

The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while the *neighborhood* of $A \subseteq V$ is $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$. By $\overline{G}$ we denote the complement of G.

A set $S \subseteq V(G)$ is *independent* (*stable*) if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the set of all the independent sets of G. An independent set of maximum cardinality will be referred to as a *maximum independent set* of G, and the *independence number* of G is $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*. If $\alpha(G) + \mu(G) = |V(G)|$, then G is called a *König-Egerváry graph* [4, 13].

Let $\Omega(G)$ denote the family of all maximum independent sets of G and

$$\begin{array}{c}\mathrm{core}(G) = \cap\{S : S \in \Omega(G)\} \text{ [10], while} \\ \mathrm{corona}(G) = \cup\{S : S \in \Omega(G)\} \text{ [3].}\end{array}$$

A set $A \subseteq V(G)$ is a *clique* in G if A is independent in $\overline{G}$, and $\omega(G) = \alpha\left(\overline{G}\right)$.

In this paper we introduce the "*Set and Collection Lemma*". It is both a generalization and strengthening of a number of elegant observations including the "*Maximum Stable Set Lemma*" due to Berge and the "*Clique Collection Lemma*" due to Hajnal.

2 Results

It is clear that the statement "*there exists a matching from a set A into a set B*" is stronger than just saying that $|A| \leq |B|$. The "*Matching Lemma*" offers both a powerful tool validating existence of matchings and its most important corresponding inequalities, emphasized in the "*Set and Collection Lemma*" and its corollaries.

Lemma 2.1 (Matching Lemma) *Let $S \in \mathrm{Ind}(G), X \in \Lambda \subseteq \Omega(G), |\Lambda| \geq 1$. Then the following assertions are true:*

(i) *there exists a matching from $S - \cap\Lambda$ into $\cup\Lambda - S$;*

(ii) *there is a matching from $S - X$ into $X - S$;*

(iii) *there exists a matching from $S \cap X - \cap\Lambda$ into $\cup\Lambda - (X \cup S)$.*

Proof. Let $B_1 = \cap\Lambda$ and $B_2 = \cup\Lambda$.

(i) In order to prove that there is a matching from $S - B_1$ into $B_2 - S$, we use Hall's Theorem, i.e., we show that for every $A \subseteq S - B_1$ we must have

$$|A| \leq |N(A) \cap B_2| = |N(A) \cap (B_2 - S)|.$$

Assume, in a way of contradiction, that Hall's condition is not satisfied. Let us choose a minimal subset $\tilde{A} \subseteq S - B_1$, for which $\left|\tilde{A}\right| > \left|N\left(\tilde{A}\right) \cap B_2\right|$.

There exists some $W \in \Lambda$ such that $\tilde{A} \nsubseteq W$, because $\tilde{A} \subseteq S - B_1$. Further, the inequality $\left|\tilde{A} \cap W\right| < \left|\tilde{A}\right|$ and the inclusion

$$N(\tilde{A} \cap W) \cap B_2 \subseteq N(\tilde{A}) \cap B_2 - W$$

imply

$$\left|\tilde{A} \cap W\right| \leq \left|N(\tilde{A} \cap W) \cap B_2\right| \leq \left|N(\tilde{A}) \cap B_2 - W\right|,$$

because we have selected $\tilde{A}$ as a minimal subset satisfying $\left|\tilde{A}\right| > \left|N\left(\tilde{A}\right) \cap B_2\right|$. Therefore,

$$\left|\tilde{A} \cap W\right| + \left|\tilde{A} - W\right| = \left|\tilde{A}\right| > \left|N(\tilde{A}) \cap B_2\right| = \left|N(\tilde{A}) \cap B_2 - W\right| + \left|N(\tilde{A}) \cap W\right|.$$

Consequently, since $\left|\tilde{A} \cap W\right| \leq \left|N(\tilde{A}) \cap B_2 - W\right|$, we infer that $\left|\tilde{A} - W\right| > \left|N(\tilde{A}) \cap W\right|$. Thus

$$\tilde{A} \cup \left(W - N(\tilde{A})\right) = W \cup \left(\tilde{A} - W\right) - \left(N(\tilde{A}) \cap W\right)$$

is an independent set of size greater than $|W| = \alpha(G)$, which is a contradiction that proves the claim.

(ii) It follows from part *(i)* for $\Lambda = \{X\}$.

(iii) By part *(i)*, there exists a matching from $S - \cap\Lambda$ into $\cup\Lambda - S$, while by part *(ii)*, there is a matching from $S - X$ into $X - S$. Since X is independent, there are no edges between

$$(S - B_1) - (S - X) = (S \cap X) - B_1 \text{ and } X - S.$$

Therefore, there exists a matching

$$\text{from } (S \cap X) - B_1 \text{ into } (B_2 - S) - (X - S) = B_2 - (X \cup S),$$

as claimed. ■

For example, let us consider the graph G from Figure 1 and $S = \{v_1, v_4, v_7\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_1, v_2, v_3, v_6, v_8, v_{10}, v_{12}\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{13}\}$. Then, there is a matching from $S - \cap\Lambda = \{v_4, v_7\}$ into $\cup\Lambda - S = \{v_2, v_3, v_6, v_8, v_{10}, v_{12}, v_{13}\}$, namely, $M = \{v_3v_4, v_7v_8\}$. In addition, we have

$$10 = 3 + 7 = |S| + \alpha(G) \leq |\cap\Lambda \cap S| + |\cup\Lambda \cup S| = 1 + 10 = 11.$$

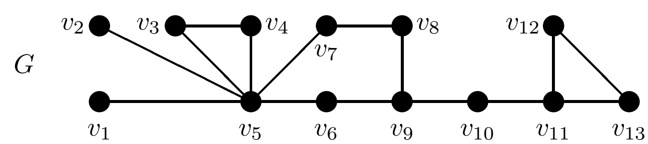

Figure 1: $\mathrm{core}(G) = \{v_1, v_2, v_{10}\}$ is not a critical set.

The assertions of Matching Lemma may be false, if the family Λ is not included in $\Omega(G)$. For instance, if $S = \{v_1, v_2, v_4, v_7, v_9, v_{12}\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_2, v_3, v_7\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$, then, there is no matching from $S - \cap\Lambda = \{v_1, v_4, v_9, v_{12}\}$ into $\cup\Lambda - S = \{v_3, v_6, v_{10}\}$. In addition, we see that

$$12 = 2 \cdot |S| \nleq |\cap\Lambda \cap S| + |\cup\Lambda \cup S| = 2 + 9 = 11.$$

Lemma 2.2 (Set and Collection Lemma) *If* $S \in \mathrm{Ind}(G)$ *and* $\Lambda \subseteq \Omega(G), |\Lambda| \geq 1$, *then*

$$|S| + \alpha(G) \leq |\cap\Lambda \cap S| + |\cup\Lambda \cup S|.$$

Proof. Let $X \in \Lambda$. By Matching Lemma *(iii)*, there is a matching from $S \cap X - \cap\Lambda$ into $\cup\Lambda - (X \cup S)$. Hence we infer that

$$\begin{aligned} |S \cap X| - |\cap\Lambda \cap S| = |S \cap X| - |\cap\Lambda \cap S \cap X| = \\ = |S \cap X - \cap\Lambda| \leq |\cup\Lambda - (X \cup S)| = \\ = |\cup\Lambda \cup (X \cup S)| - |X \cup S| = |\cup\Lambda \cup S| - |X \cup S|. \end{aligned}$$

Therefore, we obtain that

$$|S \cap X| - |\cap\Lambda \cap S| \leq |\cup\Lambda \cup S| - |X \cup S|,$$

which implies

$$|S| + \alpha(G) = |S| + |X| = |S \cap X| + |X \cup S| \leq |\cap\Lambda \cap S| + |\cup\Lambda \cup S|,$$

as claimed. ■

Corollary 2.3 *If* $\Lambda \subseteq \Omega(G), |\Lambda| \geq 1$, *then* $2 \cdot \alpha(G) \leq |\cap\Lambda| + |\cup\Lambda|$.

Proof. Let $S \in \Lambda$. By Set and Collection Lemma, we get that

$$2 \cdot \alpha(G) = |S| + \alpha(G) \leq |\cap\Lambda \cap S| + |\cup\Lambda \cup S| = |\cap\Lambda| + |\cup\Lambda|,$$

as required. ■

If $\Lambda = \Omega(G)$, then Corollary 2.3 gives the following.

Corollary 2.4 *For every graph* G, *it is true that*

$$2 \cdot \alpha(G) \leq |\mathrm{core}(G)| + |\mathrm{corona}(G)|.$$

It is clear that

$$|\mathrm{core}(G)| + |\mathrm{corona}(G)| \leq \alpha(G) + |V(G)|.$$

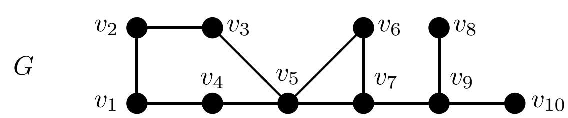

Figure 2: The graph G has $\mathrm{core}(G) = \{v_8, v_{10}\}$.

The graph G from Figure 2 has $V(G) = \mathrm{corona}(G) \cup N(\mathrm{core}(G)) \cup \{v_5\}$.

Proposition 2.5 *If* $G = (V, E)$ *is a graph with a non-empty edge set, then*

$$|\mathrm{core}(G)| + |\mathrm{corona}(G)| \leq \alpha(G) + |V| - 1.$$

Proof. Notice that for every $S \in \Omega(G)$, we have $\mathrm{core}(G) \subseteq S \subseteq \mathrm{corona}(G) \subseteq V$, which implies $\mathrm{corona}(G) - S \subseteq \mathrm{corona}(G) - \mathrm{core}(G) \subseteq V - \mathrm{core}(G)$.

Assume, to the contrary, that

$$|\mathrm{core}(G)| + |\mathrm{corona}(G)| \geq \alpha(G) + |V|.$$

Hence we infer that

$$|\mathrm{corona}(G)| - \alpha(G) \geq |V| - |\mathrm{core}(G)|,$$

i.e.,

$$|\mathrm{corona}(G) - S| \geq |V - \mathrm{core}(G)|.$$

Since $\text{corona}(G) - S \subseteq V - \text{core}(G)$, we get that $V = \text{corona}(G)$ and $\text{core}(G) = S$. It follows that $N(\text{core}(G)) = \emptyset$, since $\text{corona}(G) \cap N(\text{core}(G)) = \emptyset$.

On the other hand, G must have $N(\text{core}(G)) \neq \emptyset$, because G has a non-empty edge set and $\text{core}(G) = S \neq \emptyset$.

This contradiction proves that the inequality

$$|\text{core}(G)| + |\text{corona}(G)| \leq \alpha(G) + |V| - 1$$

is true. ■

Remark 2.6 *The complete bipartite $K_{1,n-1}$ satisfies $\alpha(K_{1,n-1}) = n - 1$, and hence*

$$|\text{core}(K_{1,n-1})| + |\text{corona}(K_{1,n-1})| = 2(n-1) = \alpha(G) + |V(K_{1,n-1})| - 1.$$

In other words, the bound in Proposition 2.5 is tight.

The graph G_1 from Figure 3 has $\alpha(G_1) = 4$, $\text{corona}(G_1) = \{v_1, v_3, v_4, v_5, v_7, v_8, v_9\}$, $\text{core}(G_1) = \{v_8, v_9\}$, and then $2 \cdot \alpha(G_1) = 8 < 2 + 7 = |\text{core}(G_1)| + |\text{corona}(G_1)|$.

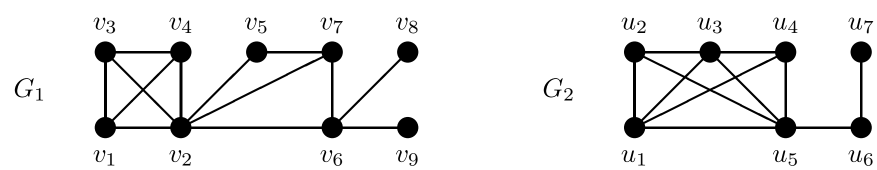

Figure 3: G_1, G_2 are non-König-Egerváry graphs.

It has been shown in [11] that

$$\alpha(G) + |\cap\{V - S : S \in \Omega(G)\}| = \mu(G) + |\text{core}(G)|$$

is satisfied by every König-Egerváry graph G, and taking into account that clearly

$$|\cap\{V - S : S \in \Omega(G)\}| = |V(G)| - |\cup\{S : S \in \Omega(G)\}|,$$

we infer that the König-Egerváry graphs enjoy the following nice property.

Proposition 2.7 *If G is a König-Egerváry graph, then*

$$2 \cdot \alpha(G) = |\text{core}(G)| + |\text{corona}(G)|.$$

It is worth mentioning that the converse of Proposition 2.7 is not true. For instance, see the graph G_2 from Figure 3, which has $\alpha(G_2) = 3$, $\text{corona}(G_2) = \{u_2, u_4, u_6, u_7\}$, $\text{core}(G_2) = \{u_2, u_4\}$, and then $2 \cdot \alpha(G) = 6 = 2 + 4 = |\text{core}(G_2)| + |\text{corona}(G_2)|$.

The *vertex covering number* of G, denoted by $\tau(G)$, is the number of vertices in a minimum vertex cover in G, that is, the size of any smallest vertex cover in G. Thus we have $\alpha(G) + \tau(G) = |V(G)|$. Since

$$|V(G)| - |\cup\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}|,$$

Corollary 2.4 implies the following.

Corollary 2.8 *[6] If* $G = (V, E)$*, then* $\alpha(G) - |\mathrm{core}(G)| \leq \tau(G) - |\cap\{V - S : S \in \Omega(G)\}|$.

Applying Matching Lemma *(i)* to $\Lambda = \Omega(G)$ we immediately obtain the following.

Corollary 2.9 *[3] For every* $S \in \Omega(G)$*, there is a matching from* $S - \mathrm{core}(G)$ *into* $\mathrm{corona}(G) - S$*.*

Since every maximum clique of G is a maximum independent set of $\overline{G}$, Corollary 2.3 is equivalent to the "*Clique Collection Lemma*" due to Hajnal.

Corollary 2.10 *[7] If* Γ *is a collection of maximum cliques in* G*, then*

$$|\cap\Gamma| \geq 2 \cdot \omega(G) - |\cup\Gamma| .$$

Another appl*ication of Matching Lem*ma is the "*Maximum Stable Set Lemma*" due to Berge.

Corollary 2.11 *[1], [2] An independent set* X *is maximum if and only if every independent set* S *disjoint from* X *can be matched into* X*.*

Proof. Matching Lemma *(ii)* is, essentially, the "*if*" part of corollary.

For the "*only if*" part we proceed as follows. According to the hypothesis, there is a matching from $S - X = S - S \cap X$ into X, in fact, into $X - S \cap X$, for each $S \in \Omega(G) - \{X\}$. Hence, we obtain

$$\alpha(G) = |S| = |S - X| + |S \cap X| \leq |X - S \cap X| + |S \cap X| = |X| \leq \alpha(G),$$

which clearly implies $X \in \Omega(G)$. ■

3 Conclusions

In this paper we have proved the "*Set and Collection Lemma*", which has been crucial in order to obtain a number of alternative proofs and/or strengthenings of some known results. Our main motivation has been the "*Clique Collection Lemma*" due to Hajnal [7]. Not only this lemma is beautiful but it is in continuous use as well. Let us only mention its two recent applications in [8, 12].

Proposition 2.7 claims that $2 \cdot \alpha(G) = |\mathrm{core}(G)| + |\mathrm{corona}(G)|$ holds for every König-Egerváry graph G. Therefore, it is true for each very well-covered graph G, [9]. Recall that G is a *very well-covered* graph if $2\alpha(G) = |V(G)|$, and all its maximal independent sets are of the same cardinality, [5]. It is worth noting that there are other graphs enjoying this equality, e.g., every graph G having a unique maximum independent set, because, in this case, $\alpha(G) = |\mathrm{core}(G)| = |\mathrm{corona}(G)|$.

Problem 3.1 *Characterize graphs satisfying* $2 \cdot \alpha(G) = |\mathrm{core}(G)| + |\mathrm{corona}(G)|$.

Let us consider a dual problem. It is clear that for every graph G there exists a collection of maximum independent sets Λ such that $2 \cdot \alpha(G) = |\cup\Lambda| + |\cap\Lambda|$. Just take $\Lambda = \{X\}$ for some maximum independent set X.

Problem 3.2 *For a given graph* G *find the cardinality of a largest collection of maximum independent sets* Λ *such that* $2 \cdot \alpha(G) = |\cup\Lambda| + |\cap\Lambda|$.

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